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SHORT COMMUNICATION

Generalized Transmuted -Generalized Rayleigh Distribution: Its Properties And Application

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Abstract

Several models have been proposed for modelling the lifetime data for reliability analysis. The performances of these models have been evaluated by fitting the models with the datasets and then compare the value with the values of the existing models. This study proposed a new model that enhance the modelling of data in reliability studies and the performance of this proposed model was evaluated by comparing it with others related models, these models include Weibull-Burr type X, Exponentiated Generalized-Burr type X and Burr type X model. The Akaike information criterion (AIC) of each model was calculated using the dataset contained about 30 units of observations. Results show that the AIC of the proposed model was found to be 10.00 while the AIC of the other three competitive related models are 192.34, 367.45 and 367.87. The result of the analysis revealed that the proposed model (Generalized Transmuted-Generalized Rayleigh) with smallest AIC had the best performance when compared to the other three competitive related models, hence the model can better be used in reliability analysis as well as in analysing skewed datasets.

Keywords: Generalized Rayleigh distribution, Hazard function, Moments

INTRODUCTION

There is always need to apply theories and distributions in solving real life problems. This sometimes, is not easily achieved. So much effort is put daily to improve distributions or to increase its flexibility. The reason of improving several lifetime distributions is to handle lifetime data and the case of real-life problems where the observed dataset does not follow any classical or standard probability model (Merovci et al. 2015).

The method of generalizing new families of distributions attract many authors to come up with different forms of such classes of distributions (Alzaatreh et al. 2013; Bourguignon et al. 2014) to give more room to the researchers to propose several distributions using these classes of distributions that can explain complex phenomenon in reliability studies, lifetime testing, engineering modelling and electronic sciences, the failure rate behaviour in the above mentioned areas can be bathtub, upside-down bathtub and others shaped but not usually monotone increasing or decreasing (Tahir and Cordeiro, 2016).

Khaleel et al (2017) introduced a four parameter continuous distribution known as exponentiated generalized burr type-X distribution. The mathematical properties of the distribution were discussed. However, the parameters were estimated using the method of maximum likelihood. Lastly, the new model was fitted to real datasets and compared the result with other models. The result showed that the exponentiated generalized burr type-X distribution provides a better fit than those it was compared with.

Merovci et al (2014) proposed a new model called the transmuted Pareto distribution. They derived expansions for the expectation, variance, moments and the moment generating function. The method of maximum likelihood was use in the estimate of parameters. An application of the transmuted Pareto distribution was demonstrated using data of exceedances of flood peak of the Wheaton River analyzed by (Choulakian and Stephens, 2011), the transmuted Pareto distribution provides better fit than the other compared distributions based on this dataset.

Oguntunde et al (2017) introduced a two parameter model called the Burr X- Exponential distribution; some few properties of the distribution were discussed. The study highlighted that the parameters can be estimated by differentiating the log-likelihood function with respect to parameters

A random variable *X* is said to follow a Generalized Rayleigh distribution, if its cumulative distribution function (CDF) and probability density function (pdf) are respectively given by Equation (1) and (2).

$$G(x; c, d) = (1-e^{-(dx)2})^{c}; x > 0, c > 0, d > 0 \quad (1)$$

and

$$g(x;c,d)=2cd^{2} xe^{-(dx)2} (-e^{-(dx)2})^{c-1}; \quad x > 0, c > 0, d > 0 \quad (2)$$

Where *c* and *d* are the shape and scale parameters respectively.

Generalized Transmuted -G family of Distribution

A random variable *X* is said to have generalized transmuted -G family of distribution if its cdf and pdf are respectively given by equations (3) and (4).

$$F (x,a,b,\lambda,\Phi) = G (x; \Phi)^{a} [(1+\lambda) -\lambda G(a, \Phi)^{b}] \quad (3)$$

and

$$f(x,a,b,\lambda,\Phi) = g(x; \Phi)^{a-1} [a(1+\lambda) -\lambda(a+b) G(a, \Phi)^{b}] \quad (4)$$

Where a, b and λare three additional positive parameters, G (x;Φ) and g(x; Φ)are the CDF and pdf of any Univariate continuous distribution defined on the parameter vector Φ. This particular distribution, generalized transmuted -G family of distribution reduces to some families of distribution studied by other authors. When a=1 and b=0, it corresponds to the Exponentiated-G family studies by (Gupta et al, 1998). If a=b=1, the GT-G family reduces to transmuted class studied by (Shaw and Buckley 2007).In addition, it reduces to the baseline distribution when a=b=1 and λ=0.

Generalized transmuted - Generalized Rayleigh distribution

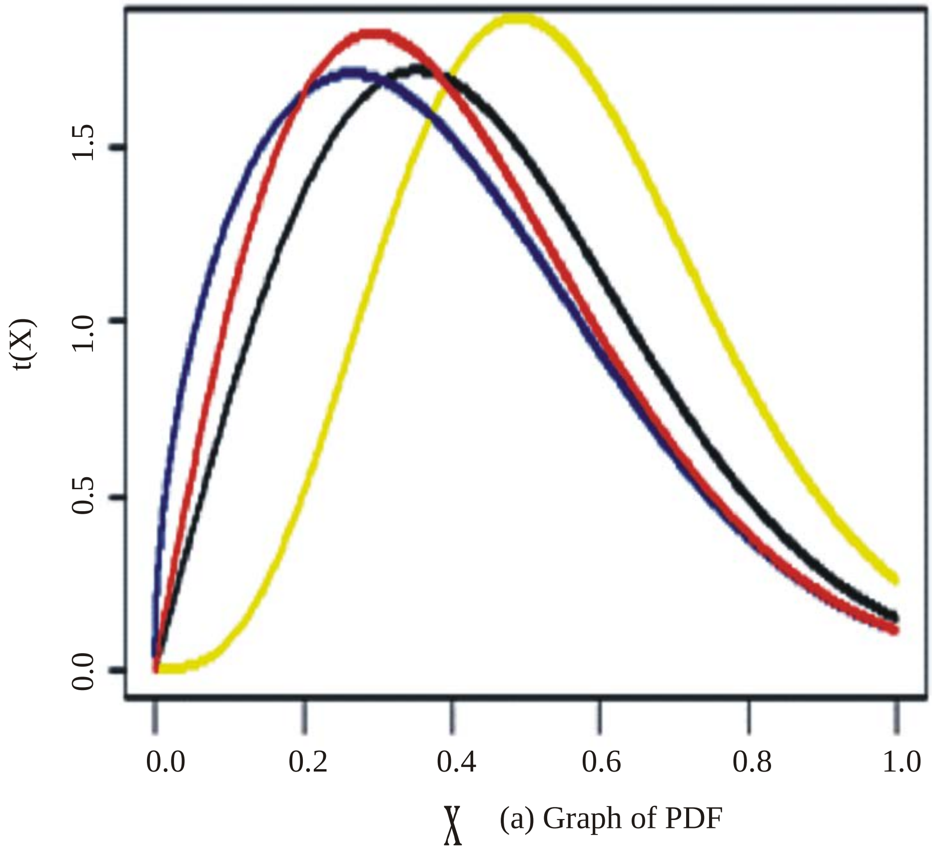
The probability density function (equation 5) of the proposed model is obtain by substituting equation (1) and (2) of the baseline in to equation (4) of the generator likewise the cumulative distribution function (equation 6) of proposed model is obtain by substituting equation (1) of the baseline in to equation (3) of the generator.

Defination1: The random variable *X* with parameters a, b, c, d and λis said to follow the generalized transmuted -generalized Rayleigh distribution if its PDF and CDF are respectively given by equation (5) and (6).

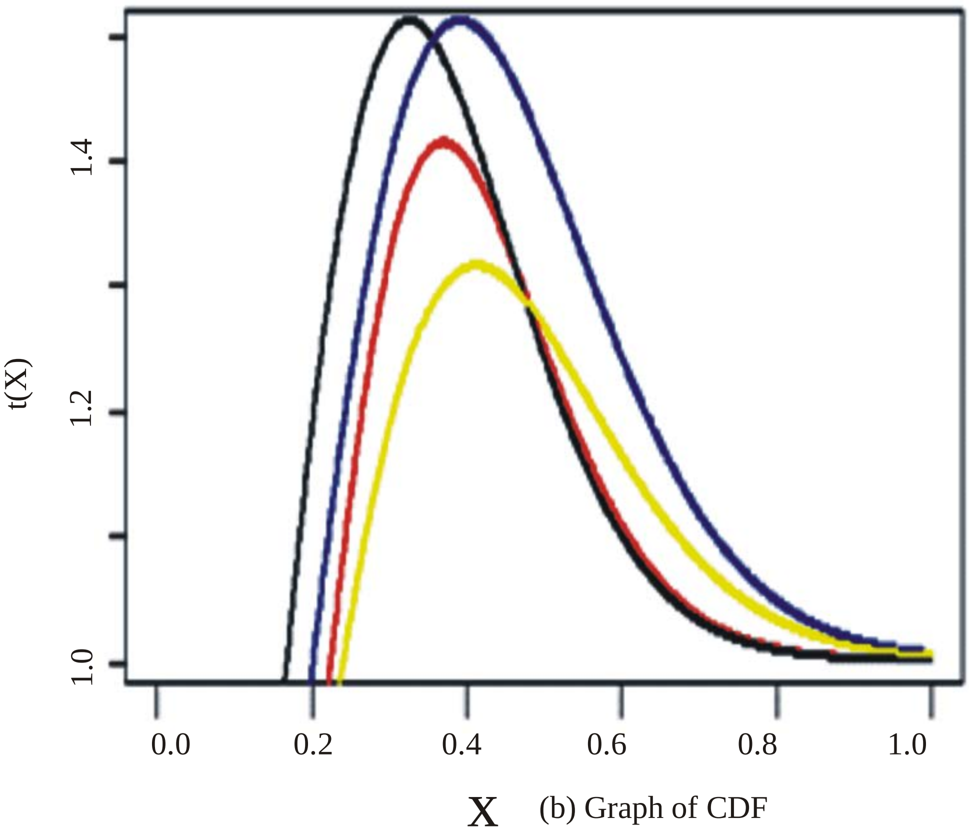
$$f(x,a,b,c,d,\lambda)=2cd^{2} xe^{-(dx)2} (1-e^{-(dx)2})^{ac-1}; [a(1+\lambda)-\lambda (a+b)(1-e^{-(dx)2})^{bc}] \quad (5)$$

and

$$f(x,a,b,c,d,\lambda)=[(1-e^{-(dx)2})]^{a};$$



(a) Graph of PDF



(b) Graph of CDF

Survival function

The survival function is the probability that a patient, device or any objects of interest will survive beyond a specified time, the survival function is also known as the reliability function. The survival function of generalized transmuted-generalized Rayleigh distribution is given by equation (7).

$$1-F(x,a,b,c,d,\lambda)=1-[(1-e^{-(dx)2})]^{a};\ [(1+\lambda)-\lambda(1-e^{-(dx)2})^{c}]^{b}] \quad (7)$$

Hazard rate Function

Hazard rate refers to the rate of death for an item of a given age, and it's also known as the failure rate. It's the likelihood that if something survives to one moment; it will also survive to the next. Hazard rate cannot be negative and only applies to those items which cannot be repaired. The hazard function of generalized transmuted-generalized Rayleigh X distribution is given by equation (8).

$$H_{GTBX}=\frac{f(x)}{1-Fx}=\frac{2cd^{2}xe^{-(dx)^{2}}(1-e^{-(dx)^{2}})^{-1}\{a(1+\lambda)-\lambda(a+b)(1-e^{-(dx)^{2}})^{bc}\}}{\{(1+\lambda)-\lambda[(1-e^{-(dx)^{2}})^{c}]^{b}\}} \quad (8)$$

Properties of the Proposed Distribution

Under this section we study some of the properties of the generalized transmuted-generalized Rayleigh distribution such as moments, moment generating function and entropy

Moments

Theorem: if X follow GTGR (a, b, c, d, λ), Then the rth moment of random variable X is given by:

$$\mu_r^{1}=\frac{c\Gamma(1+\frac{r}{2})\sum_{i=0}^{\infty}(1)^{i}}{d^{r}(1+i)^{\frac{r+2}{2}}}\left[a(1+\lambda)\binom{ac-1}{i}-\lambda(a+b)\binom{c(a+b)-1}{i}\right]$$

Proof: The rth moment of X having pdf f(x) is defined by:

$$\mu_r'=\int_0^{\infty}x^{r}f(x)dx \quad (9)$$

Substituting Equation (5) into Eq. (9) we get

$$\mu_r'=\int_0^{\infty}x^{1+r}2cd^{2}e^{-(dx)^{2}}(1-e^{-(dx)^{2}})^{ac-1}\{a(1+\lambda)-\lambda(a+b)(1-e^{-(dx)^{2}})^{bc}\}dx \quad (10)$$

$$=2acd^{2}(1+\lambda)\int_0^{\infty}x^{r+1}e^{-(dx)^{2}}(1-e^{-(dx)^{2}})^{ac-1}dx-2\lambda cd^{2}(a+b)\int_0^{\infty}x^{r+1}e^{-(dx)^{2}}(1-e^{-(dx)^{2}})^{c(a+b)-1}dx$$

But $(1-e^{-(dx)^{2}})^{ac-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{ac-1}{i}e^{-i(dx)^{2}}$ (11)

And

$$(1-e^{-(dx)^{2}})^{c(a+b)-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{c(a+b)-1}{i}e^{-i(dx)^{2}} \quad (12)$$

Substituting Equations (11) and (12) into Equation (10) we obtain

$$=\left[2acd^{2}(1+\lambda)\sum_{i=0}^{\infty}(-1)^{i}\binom{ac-1}{i}-2\lambda cd^{2}(a+b)\sum_{i=0}^{\infty}(-1)^{i}\binom{c(a+b)-1}{i}\right]\int_0^{\infty}x^{r+1}e^{-(1+i)(dx)^{2}}dx$$

Let

$y=(1+i)^{i}(dx)^{2}$ We will get,

$$\mu_r'=\left[ac(1+\lambda)\sum_{i=0}^{\infty}\frac{(-1)^{i}\binom{ac-1}{i}}{d^{r}(i+1)^{\frac{r+2}{2}}}-\lambda c(a+b)\sum_{i=0}^{\infty}\frac{(-1)^{i}\binom{c(a+b)-1}{i}}{d^{r}(i+1)^{\frac{r+2}{2}}}\right]\int_0^{\infty}y^{\frac{r}{2}}e^{-y}dy$$

And $\int_0^{\infty}y^{\frac{r}{2}}e^{-y}dy=\Gamma(1+\frac{r}{2})$ we have,

$$\mu_r' = \left[ac(1+\lambda)\sum_{i=0}^{\infty}\frac{(-1)^i\binom{ac-1}{i}}{d^r(i+1)^{\frac{r+2}{2}}}\Gamma(1+\frac{r}{2}) - \lambda c(a+b)\sum_{i=0}^{\infty}\frac{(-1)^i\binom{c(a+b)-1}{i}}{d^r(i+1)^{\frac{r+2}{2}}}\Gamma(1+\frac{r}{2})\right]$$

After simplify we have

$$\mu_r' = \frac{c\Gamma(1+\frac{r}{2})}{d^r}\sum_{i=0}^{\infty}\frac{(-1)^i}{(i+1)^{\frac{r+2}{2}}}\left[a(1+\lambda)\binom{ac-1}{i} - \lambda (a+b)\binom{c(a+b)-1}{i}\right]$$

This complete the proof

Moment generating function

The moment generation function $M_x(t)$a random variable X having pdf f(x) given in Equation (5) is obtained as:

$$M_X(t) = \int_0^{\infty} e^{tx} f(x)dx \quad (14)$$

Using power series for the exponential function we have

$$e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r \quad (15)$$

Substituting Equation (15) into Equation (14) we get

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x)dx$$

Where f(x) is the pdf of generalized transmuted-generalized Rayleigh distribution

Since $\mu' = \int_0^{\infty} x^r f(x)dx$, then $M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$

But $$\mu_r' = \frac{c\Gamma(1+\frac{r}{2})}{d^r}\sum_{i=0}^{\infty}\frac{(-1)^i}{(i+1)^{\frac{r+2}{2}}}\left[a(1+\lambda)\binom{ac-1}{i} - \lambda (a+b)\binom{c(a+b)-1}{i}\right]$$

Then the moment generating function of a random variable X follow the pdf f(x) given in Equation (5) is obtained as:

$$M_x(t) = \frac{c\Gamma(1+\frac{r}{2})}{d^r}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=0}^{\infty}\frac{(-1)^i}{(i+1)^{\frac{r+2}{2}}}\left[a(1+\lambda)\binom{ac-1}{i} - \lambda (a+b)\binom{c(a+b)-1}{i}\right]$$

The pdf of the r^{th} order statistics for a random sample X_1, X_2, X_3 ....X_n from generalized transmuted-generalized Rayleigh distribution with pdf and cdf function given by equation (5) and (6) respectively is given by:

$$f_{r,n}(x) = \frac{n!}{(n-r)!\,(r-1)} f(x)\,[F(x)]^{r-1}[1-F(x)^{n-r}$$

But $[1-F(x)]n-r = \sum_{i=0}^{n-r} (-1)I\binom{n-r}{i}F(x)i$

Then we have:

$$f_{r,n}(x) = \frac{n!}{(n-r)!\,(r-1)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i} f(x)F(x)^{r-i-1} \quad (16)$$

Substituting Equation (5) and (6) into (16) we have the rth order statistics for generalized transmuted-generalized Rayleigh distribution given by

$$f_{r,n}(x) = \frac{n!}{(n-r)!(r-1)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i} 2cd^2xe^{-(dx)^2}(1-e^{-(dx)^2})^{ac-1} \times\left[a(1+\lambda) - \lambda(a+b)(1-e^{-(dx)^2})^{bc}\right]\left[[(1-e^{-(dx)^2})^{c}]^{a}\{(1+\lambda)-\lambda[(1-e^{-(dx)^2})^{c}]^{b}\}\right]^{r+i-1} \quad (17)$$

Therefore, the pdf of the minimum order statistics $f_{1,n}(x)$ and the maximum order statistics $f_{n,n}(x)$ our proposed generalized transmuted-generalized Rayleigh distribution is given by Equations (18) and (19) respectively

$$f_{1,n}(x) = n2cd^2xe^{-(dx)^2}(1-e^{-(dx)^2})^{ac-1}\left[a(1+\lambda) - \lambda (a+b)(1-e^{-(dx)^2})^{bc}\right] \quad (18)$$

And

$$f_{n,n}(x) = n2cd^2xe^{-(dx)^2}(1-e^{-(dx)^2})^{ac-1}\left[a(1+\lambda) - \lambda(a+b)(1-e^{-(dx)^2})^{bc}\right]$$

$$\left[[(1-e^{-(dx)^2})^C]^a\{(1+\lambda)-\lambda[(1-e^{-(dx)^2})^c]^b\}\right]^{n-1} \quad (19)$$

Entropies

The Renyi entropy of a random variable X represents a measure of variation of the uncertainty. The Renyi entropy is defined by

$$I_\theta(X) = \frac{1}{1-\theta}\log\int_{-\infty}^{\infty} f(x)^\theta\,dx,\ \theta > 0 \text{ and } \theta \neq 0$$

By using the pdf in (4), we have

$$f(x)^\theta = (1+\lambda)^\theta h_a(x)^\theta \left\{1-dG(x)^b\right\}^\theta,$$

Where $d = \frac{\lambda(a+b)}{[a(1+\lambda)]}$

Then, the Renyi entropy of a random variable X having the GT-G family is given by

$$I_\theta(X) = \frac{1}{1-\theta}\log\left\{(1+\lambda)^\theta\int_{-\infty}^{\infty} h_a(x)^\theta \left\{1-dG(x)^b\right\}^\theta dx\right\}.$$

And the Renyi entropy of a random variable X having the generalized transmuted generalized Rayleigh distribution is given by

$$I_\theta(X) = \frac{1}{1-\theta}\log\left\{(1+\lambda)^\theta\left[\sum_{i,j}^{\infty}(-1)^{i+j}\binom{\theta}{i}\right]\binom{\theta(ac-1)+cbi}{j}\frac{1}{(\theta+j)^{\frac{\theta+1}{2}}}2^{\theta-1}d^{i+\theta-1}(ac)^\theta\,\Gamma\frac{\theta+1}{2}\right\}$$

The used the method of maximum likelihood estimation to estimate the parameters of the generalized transmuted-generalized Rayleigh distribution. Let $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n$ be a random sample of size n from the generalized transmuted-generalized Rayleigh distribution

With parameter vector $\Phi = (a,b,c,d,\lambda)$. Then the log-likelihood function for Φ is given by:

$$L(\Phi) = Log\prod_{i=1}^{n} f(x;\Phi) = n\log 2 + n\log c + 2n\log d + \sum_{i=1}^{n}\log x_i - d^2\sum_{i=1}^{n}\log x_i^2$$

$$(ac-1)\sum_{i=1}^{n}\log(1-e^{-(dx)^2}) + \sum_{i=1}^{n}\log\{a(1+\lambda) - \lambda(a+b)[1-e^{-(dx)^2}]^{bc}\} \quad (20)$$

The solution of the non-linear system of equations obtained by differentiating equation (20) with respect to a, b, c, d and λ gives the maximum likelihood estimates of the parameters a, b, c, d and λ.

Akaike Information Criterion (AIC)

The Akaike Information Criterion (AIC) is an estimator of the relative quality of statistical models for a given set of data. Given a collection of models for the data, AIC estimates the quality of each model, relative to each of the other models. Thus, AIC provides a means for model selection. The idea of AIC (Akaike, 1973) is to select the model that minimizes the negative likelihood penalized by the number of parameters as specified in the equation.

$$AIC = -2ll + 2k$$

Where ll denotes the log-likelihood function under the fitted model, k is the number of parameters in the model.

RESULT AND DISCUSSION

Application

In this section we compare the results of the fitted proposed model that is generalized transmuted -generalized Rayleigh distribution with others related models that are Weibull- Burr Type X, Exponentiated Generalized-Burr type X and Burr type X distribution to the dataset studied by (Meeker, 1998), the data contained about 30 units of observation. We used the Akaike information criterion AIC to evaluate the performance of our proposed model over the other related models, the models with smaller AIC is the better model.

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Table 1: The Parameters Estimates and AIC for the proposed and other related compared models. The proposed model is Generalized Transmuted- Generalized Rayleigh Distribution (GTGRD) and other related compared models are Weibull-Burr type X (WBXD), Exponentiated Generalized-Burr type X (EGBXD) and Burr type X (BXD) distributions.

Model	Maximum Likelihood Estimates.	AIC
GTGRD	$\hat{a}$=1.000, $\hat{b}$=0.9997, $\hat{c}$=1.7000, $\hat{d}$=2.5020, $\hat{\lambda}$=1.5000	10.0000
WBXD	$\hat{\alpha}$=0.0983, $\hat{\beta}$=2.0000, $\hat{\theta}$=0.4550, $\hat{\lambda}$=0.0010	192.3400
EGBXD	$\hat{\alpha}$=0.3877, $\hat{\beta}$=5.9751, $\hat{\theta}$=0.0152, $\hat{\lambda}$=0.0076	367.4500
BXD	$\hat{\alpha}$=0.4858, $\hat{\beta}$=0.0037	367.8700

Table 1 shows the summary of maximum likelihood estimates of the parameters and AIC for each fitted distribution for the dataset and we observed that the Transmuted Generalized-Generalized Rayleigh distribution with smaller AIC is a better distribution among the other three competing distributions, therefore the model can better be use in reliability analysis as well as in analysing skewed datasets.

CONCLUSION

In this paper, we proposed five parameter Generalized Transmuted-Generalized Rayleigh distributions which serve as a new distribution in the analysis of lifetime data. We derived some of the mathematical expression for this model, example pdf, cdf, survival function and hazard function. We used the method of maximum likelihood in estimating the parameters of the model in order to see how each parameter contributed toward the performance of the model. This proposed model performs better than the other competing models with smallest value of Akaike Information Criteria (AIC) equal to 10, therefore this model can appropriately fit data of various shapes that cannot be adequately fitted with existing models. Also this proposed distribution can be used in reliability analysis as well as in analysing skewed datasets.

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